LECTURES ON SINGULARITIES AND ADJOINT LINEAR SYSTEMS

LAWRENCE EIN

Abstract.

1. Singularities of Surfaces

Let (X, o) be an isolated normal surfaces singularity. The basic philosophy is to replace the singularity by a manifold. This procedure is the resolution of singularities.

Definition 1.1. Let (X, o) be a normal surface singularity. A resolution of the singularity (X, o) is a proper birational morphism $f : Y \to X$, where Y is smooth. The set $f^{-1}(o) = E_1 \cup \cdots \cup E_n$ is called the exceptional divisor.

Remark 1.2. Since X is normal, then $f^{-1}(o)$ is connected, hence has no isolated points. This implies that E_i are distinct irreducible curves.

Theorem 1.3. *The intersection matrix $\Big((E_i \cdot E_j)\Big)$ is negative definite, where E_i are the exaction curves.*

Proof. We only need to show that for any non-trivial divisor $D = \sum a_i E_i$, $D^2 < 0$. Assume for contradiction that $D^2 \geq 0$. We will reduce to that D is effective. Write $D = A - B$ such that A and B are effective and have no common components. Then $D^2 = A^2 - 2A \cdot B + B^2 \geq 0$. Since $A \cdot B \geq 0$, hence $A^2 \geq 0$ or $B^2 \geq 0$. Assume that $A^2 \geq 0$. Moreover, we may assume that X and Y are projective. Take an ample divisor H on X. Then $f^*H^2 = H^2 > 0$ and $A \cdot f^*H = 0$. By Hodge index theorem, the intersection matrix over $H^\perp$ is negative definite. This is a contradiction. $\square$

Definition 1.4. A divisor D on a projective variety X is called nef (numerically effective) if $D \cdot C \geq 0$ for any curve C on X. We say X is minimal if the canonical divisor K_X is nef.

In higher dimension, to run the MMP, we will encounter singularities. But the singularities are not too bad.

Theorem 1.5. *Let $\varphi : Y \dashrightarrow X$ be a proper birational map. If X is minimal, then φ is a morphism.*

Definition 1.6. Let $f : Y \to X$ be a proper birational morphism. We say that a divisor D is f-nef if $D \cdot C \geq 0$ for any f-exceptional curves C.

Lemma 1.7 (Negativity Lemma)**.** *Let $f : Y \to X$ be a proper birational morphism with exceptional divisor E_i. If the divisor $D = \sum a_i E_i$ is a f-nef divisor, then $-D$ is effective.*

Proof. We will first proof the case that X and Y are surface. Write $D = A - B$ where A and B are effective divisors and have no common components. Since D is f-nef, hence $D \cdot A = A^2 - A \cdot B \geq 0$. On the other hand, $A^2 \leq 0$ and $A \cdot B \geq 0$. Therefore, $A^2 - A \cdot B \leq 0$. This implies that $A = 0$. Therefore $-D = B$ is effective.

For higher dimensional case, we may cut X by $n - 3$ general hypersurface to reduce to the surface case. ☐

Proof of the Theorem. Take a resolution of the birational map φ. We get the a variety Z and two birational morphisms $f : Z \to Y$ and $g : Z \to X$. We will use the nefness of K_X to show that no such resolution is needed. Write

$$\begin{aligned} K_Z &= f^* K_Y + \sum a_i E_i + \sum f_j F_i \\ &= g^* K_X + \sum a_i' E_i + \sum g_k G_k, \end{aligned}$$

where E_i are f- and g-exceptional divisors, F_j are f-exceptional but not g-exceptional divisors and G_k are g-exceptional but not f-exceptional divisor. Then a_i, a'_i, f_j and g_k are nonnegative. Consider the difference

$$(g^*K_X + \sum a'_iE_i + \sum g_kG_k) - (f^*K_Y + \sum a_iE_i + \sum f_jF_i) = 0.$$

We get

$$g^*K_X - f^*K_Y + \sum g_kG_k = \sum f_jF_i + \sum (a_i - a'_i)E_i.$$

Note that the left hand side is f-nef since K_X is nef and G_k are not f-exceptional. Then by Negativity Lemma, $-\sum f_jF_i + \sum(a'_i - a_i)E_i$ must be nef. Hence f_j must be zero and $a'_i \geq a_i$. In other words, there is no f-exceptional but not g-exceptional divisors. Hence, there is no blowing up of φ needed. Therefore, $\varphi : Y \to X$ is a morphism. □

Let X be a smooth projective variety. Denote by $N_1(X)_{\mathbb{R}}$ the space of 1-cycles of X modulo numerical equivalence. It is know that $N_1(X)$ is a finite dimensional space. Denote by $NE_1(X)$ the subspace of effective curves.

Theorem 1.8 (Cone Theorem)**.**

$$\overline{NE}_1(X) = \overline{NE}_1(X)_{K_X \geq 0} + \sum \mathbb{R}_+ C_i,$$

where C_i are rational curves such that $-K_X \cdot C_i \leq \dim X + 1$. Moreover, C_i's form a countable collection.

Those C_i's a

Assume that X is a surface. Then

(1) if $-K_X \cdot C_i = 1$ then $C_i^2 = -1$ which implies that C_i is a smooth (-1)-curve.

(2) if $-K_X \cdot C_i = 2$ then $C_i^2 = 0$ which implies that X is a ruled surface.

(3) if $-K_X \cdot C_i = 3$ then $X = \mathbb{P}^2$.

Definition 1.9. Let $f : Y^2 \to (X^2, o)$ be a resolution of an isolated normal surface singularity and $f^{-1}(o) = E_1 \cup \cdots \cup E_n$. A effective cycle $Z_{nef} = \sum a_i E_i$ is called a minimal nef cycle, if $-Z_{nef}$ is nef and $Z_{nef} \leq D$ for any effective cycle $D = \sum d_i E_i$ such that $-D$ is nef.

Proposition 1.10. *Z_{nef} is well-defined and unique.*

Proof. Let $D = \sum d_i E_i$ and $D' = \sum d'_i E_i$ be two effective cycles such that $-D$ and $-D'$ are nef. Define $D'' = Min(D, D') = \sum \min(d_i, d'_i) E_i$. We claim that $-D''$ is nef. Write $D = D'' + R_1$ and $D' = D'' + R_2$. Then R_1 and R_2 have no common components. Therefore any exceptional curve E_i can appear in at most one of the two cycles R_1 and R_2. Without lose of generality, we assume that E_i does not appear in R_1. Then $D'' \cdot E_i = D \cdot E_i - R_1 \cdot E_i \leq 0$. Therefore $-D''$ is nef. $\square$

1.1. Rational singularities.

Definition 1.11 (Rational Singularity)**.** A morphism $f : Y \to X$ is said to be a rational resolution if Y is smooth and f is a proper and birational morphism such that $R^i f_* \mathscr{O}_Y = 0$ for $i > 0$.

Proposition 1.12. *Let $f : Y \to X$ be a rational resolution and $f' : Y' \to X$ be another resolution. Then $f' : Y' \to X$ is also a rational resolution.*

Proof. We have a birational map $\varphi : Y \dashrightarrow Y'$. Successively Blowing up the undefined locus of φ, we get a variety Z and two proper birational morphisms $g : Z \to Y$ and $g' : Z \to Y'$ such that $h := f \circ g = f' \circ g'$. Since g is the composition of blowing-ups. Then $R^q g_*(\mathscr{O}_Z) = 0$ for $q > 0$. Apply the Leray spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathscr{F})) \Rightarrow R^{p+q}(f \circ g)_*(\mathscr{F}).$$

It follows that $R^i h_* \mathscr{O}_Z = 0$ for $i > 0$. Apply the Leray spectral sequence to $f' \circ g'$. It is easy to see that $R^1 f'_* \mathscr{O}_Y = 0$. In fact, it fits in the following exact sequence

$$0 \to R^1 f'_* \mathscr{O}_Y \to R^1 h_* \mathscr{O}_Z \to f'_* R^1 g' \mathscr{O}_Z.$$

Since Y' is smooth hence Y' has a rational resolution. Now Z is another resolution of Y'. By the above argument, we can conclude that

$R^1 g'_* \mathscr{O}_Z = 0$. Apply the Leray spectral sequence to $p+q=2$. We see that $R^2 f'_* \mathscr{O}_Y = 0$. Hence $R^2 g'_* \mathscr{O}_Z = 0$. By induction, we conclude that $R^p f'_* \mathscr{O}_Y = 0$ for $p > 0$. $\square$

This shows that rational resolution is well-defined.

Proposition 1.13. *Let $f : Y \to (X, o)$ be a resolution of a rational surface singularity. Then $\chi(\mathscr{O}_D) \leq 1$.*

Proof. Since (X, o) is rational and normal, then $H^1(\mathscr{O}_D) = H^1(\mathscr{O}_Y) = 0$. Therefore, $\chi(\mathscr{O}_D) \leq 1$. $\square$

What if $R^i f_* \mathscr{O}_Y \neq 0$?

Let $f : Y \to (X, o)$ be a resolution. Then $R^1 f_* \mathscr{O}_Y$ is a finite length module supported at the origin. It is also an invariant of the singularity, called the geometric genus of o (See Kollár). Since smooth varieties has rational resolution, then $R^1 g'_* \mathscr{O}_Z = 0$. Therefore, the Leray spectral sequence tells us that $R^1 f_* \mathscr{O}_Y = R^1 f'_* \mathscr{O}_Y$.

Reference: Miled Reid, Park city Lecture notes.

Since $R^1 f_* \mathscr{O}_Y$ is supported at the origin, by Serre-Grothendieck spectral sequence, we know that $H^1(\mathscr{O}_Y) = R^1 f_* \mathscr{O}_Y$. To compute higher direct image sheaves, besides the definition, we have the formal function theorem.

Theorem 1.14. *Let $f : Y \to X$ be a proper morphism and S be a subvariety of X. Then for any coherent sheaf $\mathscr{F}$ on Y, we have*

$$\widehat{R^p f_* \mathscr{F}} = \varprojlim_k R^p f_* (\mathscr{F} / I^k \mathscr{F}),$$

where I is the defining ideal of S in Y and the left hand side is the completion along $I\mathscr{O}_Y$.

Remark 1.15. The left hand side in fact is isomorphic to $R^p f_* \mathscr{F}$ since $R^p f_* \mathscr{F}$ is coherent. A completion of a finitely generated module M

over a Noetherian ring R can be obtained by extension of scalars: $\hat{M} = M \otimes_R \hat{R}$.

By the formal function theorem, we see that there is an isomorphism

$$H^1(O_Y) = \varprojlim_{\mathrm{Supp}(D)\subset f^{-1}(o)} H^1(\mathscr{O}_D).$$

On the other hand, we have morphism $H^1(\mathscr{O}_Y) \to H^1(\mathscr{O}_D)$. Therefore, there is a D such that $H^1(\mathscr{O}_Y) \cong H^1(\mathscr{O}'_D)$ for all $D' \geq D$.

Definition 1.16 (Cohomology cycle)**.** Let $f : Y \to (X, 0)$ be a resolution of an isolates normal surface singularity (X, o). A divisor D supported on the exceptional locus is called a cohomology cycle if $H^1(\mathscr{O}_Y) \cong H^1(\mathscr{O}_D)$.

Theorem 1.17. *There exists a unique minimal cohomology cycle.*

The proof is similar to the proof of existence of minimal nef cycle.

Proof. Let $D = \sum d_i E_i$ and $D' = \sum d'_i E_i$ be two cohomology cycles. We claim that $D'' = \min(D, D') = \sum \min d_i, d'_i E_i$ is also a cohomology cycle. Write $D = D'' + R_1$ and $D' = D'' + R_2$. Then R_1 and R_2 have no common components. Therefore, $I_{D\cap D'} = \mathscr{O}_X(-D'') \otimes I_\Sigma$, where $\Sigma = R_1 \cap R_2$.

We have the following exact sequences.

$$0 \to I_{D\cup D'} \to I_D \oplus I_{D'} \to I_{D\cap D'} \to 0,$$

which induces an exact sequence

$$0 \to \mathscr{O}_{D\cup D'} \to \mathscr{O}_D \oplus I_{D'} \to \mathscr{O}_{D\cap D'} \to 0.$$

Therefore, we have an exact sequence

$$H^1(\mathscr{O}_{D\cup D'}) \to H^1(\mathscr{O}_D) \oplus H^1(\mathscr{O}_{D'}) \to H^1(\mathscr{O}_{D\cap D'}) \to 0.$$

It is also easy to check that

$$0 \to \mathscr{O}_\Sigma(-D'') \to \mathscr{O}_{D\cap D'} \to \mathscr{O}_{D''} \to 0$$

is exact by snake lemma. Therefore, $h^1(\mathscr{O}_{D''}) \geq h^1(\mathscr{O}_D) + h^1(\mathscr{O}_{D'}) - h^1(\mathscr{O}_{D\cap D'}) = h^1(\mathscr{O}_Y)$. We conclude that $H^1(\mathscr{O}_Y) \to H^1(\mathscr{O}_{D''})$ is an isomorphism. □

Let $f : Y \to (X, o)$ be a resolution of a rational surface singularity. Assume that X is affine. Then $H^1(\mathscr{O}_Y) = 0$. Moreover, $H^2(\mathscr{O}_Y) = 0$. From the exponential seque

$$0 \to \mathbb{Z} \to \mathscr{O}_Y \to \mathscr{O}_Y^* \to 0$$

we see that

$$\operatorname{Pic}(Y) = H^1(\mathscr{O}_Y^*) = H^2(Y, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot L_i,$$

where L_i are divisors such that $L_i \cdot E_j = \delta_{ij}$. A divisor L on Y is then determined by the intersection numbers $a_i = L \cdot E_i$. Slightly move L_i to L_i', we get the same intersection number. Hence $\mathscr{O}_Y(L_i)$ is globally generated. A divisor L on Y is nef if and only if $a_i \geq 0$. Line bundles associated to nef divisors are globally generated. The nef cone $\operatorname{Nef}(Y)$ is $\sum \mathbb{Z}_+ L_i$.

Let $f : Y \to (X, o)$ be a resolution of an isolated surface singularity. Denote the maximal ideal associated to o by $\mathfrak{m}$. Then $\mathfrak{m}\mathscr{O}_Y = \mathscr{O}_Y(-Z_{max}) \otimes I_\Sigma$ where $\dim \Sigma = 0$. Morevore, $-Z_{max}$ is nef and $Z_{max} \geq Z_{nef}$.

Definition 1.18 (Fundamental cycle)**.** The cycle Z_{max} is called the fundamental cyle.

Remark 1.19. In Miles Reid's book, Z_{max} is called the fiber cycle.

Theorem 1.20 (Artin)**.** *Assume that (X, o) is an rational surface singularity. Then*

(1) $\mathfrak{m}\mathscr{O}_Y = \mathscr{O}_Y(-Z_{max})$, *i.e.* $I_\Sigma = \mathscr{O}_Y$. *Moreover,* $Z_{max} = Z_{nef}$.

(2) $\mathfrak{m}/\mathfrak{m}^2 = H^0(\mathscr{O}_{Z_{max}}(-Z_{max})$ *and the embedding dimension of o is* $\dim \mathfrak{m}/\mathfrak{m}^2 = -Z_{max}^2 + 1$.

(3) The multiplicity of o is $-Z_{max}^2$.

Proof. We may assume that X is affine. Since (X,o) is rational and $-Z_{max}$ is nef, then $\mathscr{O}_Y(-Z_{max})$ is globally generated, equivalently, $H^0(Y,\mathscr{O}_Y(-Z_{max}))\otimes\mathscr{O}_Y\to\mathscr{O}_Y(-Z_{max})$ is surjective. Notice that $\mathfrak{m}=H^0(Y,\mathscr{O}_Y(-Z_{max}))$ and $\mathfrak{m}\mathscr{O}_Y\subset\mathscr{O}_Y(-Z_{max})$. Therefore, $\mathfrak{m}\mathscr{O}_Y=\mathscr{O}_Y(-Z_{max})$. Since $Z_{max}\geq Z_{nef}$, hence $\mathfrak{m}=H^0(\mathscr{O}_Y(-Z_{max}))\subset H^0(\mathscr{O}_Y(-Z_{nef}))$. Therefore $H^0(\mathscr{O}_Y(-Z_{nef}))=\mathfrak{m}$. Since $\mathscr{O}_Y(-Z_{nef})$ is also globally generated, then $\mathfrak{m}\mathscr{O}_Y=\mathscr{O}_Y(-Z_{nef})=\mathscr{O}_Y(-Z_{max})$. Take the cohomology of the following short exact sequence

$$0\to\mathscr{O}_Y(-2Z_{max})\to\mathscr{O}_Y(-Z_{max})\to\mathscr{O}_{Z_{max}}(-Z_{max})\to 0.$$

Note that $H^1(\mathscr{O}_Y(-2Z_{max})=0$ and $H^1(\mathscr{O}_{Z_{max}}(-Z_{max})=0$ because $-Z_{max}$ is nef and big. We then have the following short exact sequence

$$0\to\mathfrak{m}^2=H^0(\mathscr{O}_Y(-2Z_{max})\to\mathfrak{m}=H^0(\mathscr{O}_Y(-Z_{max})\to H^0(\mathscr{O}_{Z_{max}}(-Z_{max}))\to 0.$$

The embedding dimension is given by

$$h^0(\mathscr{O}_{Z_{max}}(-Z_{max}))=\chi(\mathscr{O}_{Z_{max}}(-Z_{max}))=-Z_{max}^2+1-p_a(\mathscr{O}_{Z_{max}})=-Z_{max}^2+1.$$

Apply the same trick to $\mathfrak{m}^k/\mathfrak{m}^{k+1}$, we see that

$$\dim\mathfrak{m}^k/\mathfrak{m}^{k+1}=h^0(\mathscr{O}_{Z_{max}}(-kZ_{max}))=-kZ_{max}^2+1.$$

Hence $\mathrm{mult}_oX=-Z_{max}^2$. □

By Artin's theorem, we know that a rational surface singularity is a hypersurface singularity if $Z_{max}^2=-2$.

1.2. **Du Val singularities.**

Definition 1.21. Let $f:Y\to(X,o)$ be a minimal resolution of an isolated surface singularity. Denote the fundamental cycle by Z. We say (X,o) is a Du Val singularity if $K_Y\cdot E=0$ for any exceptional curve E.

Theorem 1.22. *An isolated surface singularity (X,o) is Du Val if and only if (X,o) is a rational double point, equivalently a rational singularity such that $Z^2=-2$.*

Proof. Assume that (X, o) is a rational double point. Then $K_Y \cdot Z = 2p_a(Z) - 2 - Z^2 = -2\chi(\mathscr{O}_Z) - Z^2 = 0$. Because Y is minimal. We know that K_Y is nef. As the fundamental cycle, Z supported on all exceptional curves. Therefore, $K_Y \cdot E_i = 0$.

Assume that (X, o) is a Du Val singularity. We need to show that $R^1 f_* \mathscr{O}_Y = 0$. By formal function theorem, it suffices to show that $H^1(\mathscr{O}_D) = 0$ for all effective D supported on exceptional curves. We do this by induction. Write $\mathscr{O}_Y = \mathscr{O}_Y(K_Y + A)$. Then $A \cdot E = 0$ for any exceptional curves E. Assume that $D = E$ is irreducible. Then $\mathscr{O}_D = (K_Y + A)|_D = K_D + (A - E)|)D$. Note that $\deg((A - E)|)D) = -D^2 \geq 0$. Therefore,

$$H^1(\mathscr{O}_D) = H^1(K_E + (A - E)|_E) = H^0(-(A - E)|_E) = 0.$$

Now write $D = D' + E$. $\square$

The classification of Du Val singularities are well known. The following are all the possible Du Val singularities.

$$\begin{aligned}
A_n : &\quad x^2 + y^2 + z^{n+1} = 0, \\
D_n : &\quad x^2 + y^2 z + z^{n-1} = 0, \\
E_6 : &\quad x^2 + y^3 + z^4 = 0, \\
E_7 : &\quad x^2 + y^3 + yz^3 = 0, \\
E_8 : &\quad x^2 + y^3 + z^5 = 0.
\end{aligned}$$

1.3. **Canonical model.** Let X be a minimal surface of general type. Then K_X is nef and $K_X^2 > 0$. By Reider's theorem, which will be discussed later, $X_{can} = \mathrm{Proj} \oplus H^0(mK_X)$ is a surface. There is a canonical birational morphism $X \to X_{can}$. The curves being contracted are the curves C such that $C\dot{K}_X = 0$. Therefore, X_{can} has only Du Val singularities.

1.4. **Gorenstein Singularities.** Let X be a normal surface. We can a canonical sheaf ω_X by extending the canonical bundle $\mathscr{O}_U(K_U)$, where

U is the smooth locus of X. A normal variety X is Gorenstein if ω_X is a line bundle.

Example 1.23. Let S be the surface $x^2 + y^2 + z^2 = 0$. Then $\omega_S \cong \mathscr{O}_S(K_{\mathbb{A}^3}|_S) \otimes \mathscr{N}_S$.

Assume that $f : Y \to (X, o)$ is a minimal resolution. Since the intersection matrix is negative definite, there is a unique $\mathbb{Q}$-cycle $Z_K = \sum a_i E_i$ such that $K_Y \cdot E_i = -Z_K \cdot E_i$ for any exceptional curve E_i. By negativity Lemma, Z_K is effective.

Proposition 1.24. *Assume that (X, o) is a normal Gorenstein surface singularity. Let $f : Y \to (X, o)$ be the minimal resolution.*

(1) *K_Y is linear equivalent to an integral cycle $-Z_K = \sum a_i E_i$ where $a_i \geq 0$. Moreover, if $\mathscr{O}_{K_Y}$ is nontrivial, then $a_i > 0$.*

(2) *$Z_K = Z_{coh}$.*

Proof. (1) Choose an effective cycle $G = D + \sum b_i E_i \sim K_Y$, where the components of D are not f-exceptional. Then $f_* G = \bar{D} \sim K_X \sim 0$. Now note that $f^* \bar{D} \sim 0$. So $K_Y \sim G - f^* f_* G = -\sum a_i E_i$. We see that K_Y is linearly equivalent to $-\sum a_i' E_i$. We know that K_Y is also numerical equivalent to $-\sum a_i E_i$. Since $\Big(E_i \cdot E_j\Big)$ is negative definite. So $a_i' = a_i$ and $K_Y \sim -Z_K$.

(2) Now $\mathscr{O}_Y(-\sum a_i E_i) = \mathscr{O}_Y(-Z_K) = \mathscr{O}_Y(K_Y)$. Consider the short exact sequence

$$0 \to \mathscr{O}_Y(-Z_K) \to \mathscr{O}_Y \to \mathscr{O}_{Z_K} \to 0.$$

By the Grauert-Riemannschneider Theorem, $R^i f_* \mathscr{O}_Y(K_Y) = 0$ for all $i > 0$. Therefore, $H^1(\mathscr{O}_Y) = R^1 f_* \mathscr{O}_Y \to R^1 f_* \mathscr{O}_Z = H^1(\mathscr{O}_Z)$ is an isomorphism. So Z_K is a cohomology cycle. Since Z_{con} is the unique minimal cohomology cycle, then $Z_{coh} \subseteq Z_K$. If $Z_K - Z_{con}$ is strictly effective, then we may write $Z_K = E + D$ such that E is an irreducible component and $Z_{coh} \subseteq D$. So

$H^1(\mathscr{O}_D) \cong H^1(\mathscr{O}_Y)$. We will show that this can not be the case. Consider the exact sequence

$$0 \to \mathscr{O}_E(-D) \to \mathscr{O}_{Z_K} \to \mathscr{O}_D \to 0.$$

We want to show that $H^1(\omega_{Z_K}) \to H^1(\mathscr{O}_D)$ has non-trivial kernel. By Serre duality, $H^1(\mathscr{O}_{Z_K}) = H^0(\omega_{Z_K})$ and $H^1(\mathscr{O}_D) = H^0(\omega_D)$. Note that $\omega_{Z_K} = (K_Y + Z_K)|_{Z_K} = \mathscr{O}_{Z_K}$ and $\omega_D = (K_Y + D)|_D = \mathscr{O}_D(-E)$. Now consider the exact sequence,

$$0 \to \mathscr{O}_D(-E) \to \mathscr{O}_{Z_K} \to \mathscr{O}_E \to 0.$$

We see easily that $H^0(\omega_D) \to H^0(\omega_{Z_K})$ is a nonzero map. This completes the argument.

□

Theorem 1.25. *Assume that $f : Y \to (X, o)$ be a minimal resolution of a normal surface singularity. If K_Y is numerically equivalent to $-Z_K = \sum a_i E_i$ where $a_i \in \mathbb{Z}^+$ and $Z_K = Z_{con}$, then (X, o) is a Gorenstein singularity.*

Lemma 1.26. $\mathrm{Pic}(Z_K) \cong \mathrm{Pic}(Y)$

Proof. Note that $\mathrm{Supp}(Z_K)$ is a deformation retract of Y. Apply five-lemma to cohomolgy sequences of the exponential sequences of Y and Z_K, we get the isomorphism. □

Lemma 1.27. *$K_Y \cong \mathscr{O}_Y(-Z_K)$.*

Proof. By our assumption, $L = K_Y + Z_K$ is numerically trivial. We only need to show that $L|_Z = \omega_Z$ is trivial. Then by the previous lemma L is trivial. We will show that $L|_Z$ is globally generated. If so, we will have a morphism $\mathscr{O}_Z \to L|_Z$. Since $L|_Z$ is numerically trivial, then $\mathscr{O}_Z \to L|_Z$ must be an isomorphism. Write $Z_K = E + D$, where E is an irreducible component of Z_K. By applying snake lemma, we get the following exact sequence

$$0 \to \mathscr{O}_Y(K_Y + D)|_D \to L|_Z \to \omega_Z|_E \to 0.$$

Since Z_K is the minimal cohomology cycle, then $H^1(\mathscr{O}_E(-D)) \to H^1(\mathscr{O}_{Z_K})$ is a non-zero map. By Serre duality, $H^0(\mathscr{O}_{Z_K}) \to H^0(\omega_{Z_K}|_E)$ have a non-zero map. Since ω_Z is numerically trivial. So $\omega_{Z_K}|_E = \mathscr{O}_E$ and the section 1 of $H^0(\mathscr{O}_E)$ can be lifted to ω_{Z_K}. This shows that ω_{Z_K} is generated by the section. We conclude that $\omega_{Z_K} = \mathscr{O}_{Z_K}$. $\square$

2. Flip constructed from linear algebra

Let $V = \mathbb{C}^{a+1}$ and $W = \mathbb{C}^{b+1}$ with $b \geq a \geq 0$. Set $H = \mathrm{Hom}(V, W) = \mathbb{C}^{(a+1)(b+1)}$. Consider the subvariety $X = \{\varphi \in H \mid \mathrm{rank}\varphi \leq 1\}$ of H. Outside the origin, actions of $GL(V)$ and $GL(W)$ will move $\varphi \in X$ transitively. Therefore, X has an isolated singularity at the origin. We will see that (X, o) has two natural resolutions.

One way to say that $\varphi \in X$ is rank one is a 1-dimensional quotient space $\mathrm{Im}(\varphi) = V/U$ of some a-dimensional subspace $U \subset W$. Another way is that $\mathrm{Im}(\varphi)$ is a 1-dimensional subspace of W.

Denote by $P = \mathbb{P}^a = \mathbb{P}(V)$. 1-dimensional quotients of V are classified by $\mathscr{O}_P(1)$. Let $Y = \mathscr{H}om(\mathscr{O}_P(1), W \otimes \mathscr{O}_P) = W \otimes \mathscr{O}_P(-1)$. As the total space of a vector bundle Y is smooth. We have a natural morphism $f : Y \to X \subset H$, which maps a morphism $\mathscr{O}_P(1) \to W \otimes \mathscr{O}_P$ to the composition $V \otimes \mathscr{O}_P \to \mathscr{O}_P(1) \to W \otimes \mathscr{O}_P$. It is easy to see that the pre-image of a point outside the zero point of X is a single point. Hence $Y \to X$ is a resolution. In fact, $Y \to X$ collapse the zero section of Y. Denote the zero section by Z. Then the normal bundle $\mathscr{N}_Z$ is isomorphic to $W \otimes \mathscr{O}_P(-1)$. Therefore, $\Omega_{Y/P} \cong \mathscr{N}_Z^* = W^* \otimes \mathscr{O}_P$ and $K_Y|_Z \cong \mathscr{O}_P(-a-1)$. Hence $K_Y = \pi^* K_P \otimes K_{Y/P} \cong \pi^* \mathscr{O}_P(b-a)$ which is nef by our assumption.

Denote by $Q = \mathbb{P}^b = \mathbb{P}(W^*)$ and $Y' = \mathscr{H}om(V \otimes \mathscr{O}_Q, \mathscr{O}_Q(-1)) = V \otimes \mathscr{O}_Q(-1)$. Then Y' is also a resolution which collapse the zero section of Y'. The morphism $Y' \to X$ sent a morphism $V \otimes \mathscr{O}_Q \to \mathscr{O}_Q(-1)$ to the compostions $V \otimes \mathscr{O}_Q \to \mathscr{O}_Q(-1) \to W \otimes \mathscr{O}_P$. Similarly, we get $K_{Y'} \cong \pi^* \mathscr{O}_Q(a-b)$ which is not nef.

This is an example of flips. Moreover, X has a rational singularity at the origin. Denote $T = H \times \mathbb{P}(W)$. T can be viewed as a trivial vector bundle over $\mathbb{P}(W)$ or a projective bundle of the trivial vector bundle $W \otimes \mathscr{O}_H$ over H. Let $p : T \to H$ and $q : T \to \mathbb{P}(W)$ be the two projections. Y is the vector bundle $V \otimes \mathscr{O}_{\mathbb{P}(W)}(-1)$. Consider the universal maps $p^*V \otimes \mathscr{O}_T \to p^*W \otimes \mathscr{O}_T$ on T. We also have the following exact sequence on T

$$p^*W \otimes \mathscr{O}_T \to \mathscr{O}_T(1) \to 0.$$

Therefore, V can be identified as the zero locus of a section of $p^*V \otimes \mathscr{O}_T(-1)$. This will give a resolution of $\mathscr{O}_Y$. By chasing the resolution, we can prove that $R^i p_* \mathscr{O}_Y = 0$ for $i > 0$.

3. Singularities of theta divisors

Let C be a smooth projective curve of genus g. We denote the d-th symmetric product of C by $C^{(d)}$. Let $\mathrm{Pic}^d(C)$ be the degree d component of the Picard group $\mathrm{Pic}(C)$. Set $W^r_d = \{[L] \in \mathrm{Pic}^d(C) \mid h^0(L) > r\}$. There is a morphism

$$\varphi : C^{(d)} \to W^0_d \subset \mathrm{Pic}^d(C)$$

$$D = \sum_{i=1}^{d} p_i \in C^{(d)} \mapsto [\mathscr{O}_C(D)].$$

Assume that L is a line bundle on C such that $r = h^0(L) > 0$. The fiber

$$\varphi^{-1}([L]) = \{D \mid D \geq 0 \text{ and } \mathscr{O}_X(D) \sim L\} = \mathbb{P}(H^0(L)^*) \cong \mathbb{P}^{r-1}.$$

When $\deg L = g - 1$, $\chi(L) = h^0(L) - h^1(L) = \deg L + 1 - g = 0$. So $h^0(L) = h^0(K_C - L)$. Denote by $\Theta = \varphi(C^{(g-1)})$ which is called the theta divisor. For general L, $h^0(L) = h^1(L) = 1$. (??? Why?). This says that $\varphi : C^{(g-1)} \to \Theta$ is birational.

Theorem 3.1 (Mumford)**.**

$$\dim W^r_{g-1} \leq (g-1) - 2r - 1.$$

Let $\Sigma = \{x \in C^{(g-1)} \mid \dim \varphi^{-1}(\varphi(x)) \geq 1\}$. By Mumford's theorem, we know that that $\dim \Sigma \leq g-3$. Hence φ is a small resolution.

The theta divisor is not smooth in general.

Theorem 3.2 (Riemann)**.**

$$\mathrm{mult}_{[L]}\Theta = \dim H^0(L).$$

Riemann singularity theorem tells us Θ has the singular locus $W^1_{(g-1)}$.

However, singularities of Θ are not to bad. The following theorem tells us the Θ has rational singularities.

Theorem 3.3 (Kempf)**.** *The theta divisor Θ has rational singularities.*

The proof is not very difficult. We will use the fact that Θ is a hypersurface.

Proof. We want to show that $R^i\varphi_*\mathscr{O}_{C^{(g-1)}} = 0$ for $i > 0$. Since Θ is a hypersurface, then K_Θ is a line bundle. Because φ is a small resolution, we see that $K_{C^{(g-1)}} = \varphi^*K_\Theta$. By Grauert-Riemannshneider theorem and projective formula, we see that

$$R^i\varphi_*\mathscr{O}_{C^{(g-1)}} = R^i\varphi_*(K_{C^{(g-1)}} \otimes \varphi^*(K_\Theta^{-1})) = R^i\varphi_*K_{C^{(g-1)}} \otimes K_\Theta^{-1} = 0.$$

$\square$

Using the fact that Θ is a hypersurface, in fact a determinantal variety, at a point $[L]$, Θ can be defined by a polynomial $f = f_m + f_{m+1} + \cdots$, where f_m is a degree m homogenious polynomial. Blowing up $[L]$, the tangent cone is define by f_m in $\mathbb{P}^{g-1}$. Let $\Lambda = \varphi^{-1}([L]) \cong \mathbb{P}^r \subset C^{(g-1)}$ and E be the exceptional divisor of the blowing up $\psi : Bl_\Lambda C^{(g-1)} \to C^{(g-1)}$. We know that φ is a rational resolution, so is the composition $\varphi \circ \psi$. Therefore, $\dim_k \mathfrak{m}^{t+1}/\mathfrak{m}^t = h^0(\mathscr{O}_E(-tE))$, where $\mathfrak{f}$ is the maximal ideal of $[L]$. So we get $\mathrm{mult}_{[L]}\Theta = c_1(\mathscr{O}_E(-E))^{g-2}$. Note that the exceptional divisor E is the projectivization of the normal bundle $N = N_{\Lambda/C^{(g-1)}}$. So $\mathscr{O}_E(-E) = \mathscr{O}_{\mathbb{P}(N)}(1)$. On $\mathbb{P}(N)$, we have the universal sequence

$$0 \to \Omega^1_{N/\Lambda}(1) \to \psi^* N^* \to \mathscr{O}_P(1) \to 0.$$

So $c_1(\mathscr{O}_P(1)) = \psi^*(c_1(N^*))$ and $c_1(\mathscr{O}_E(-E))^{g-2} = \psi^*(c_1(N^*))^{(g-2)}$. Now we compute the normal bundle N. Consider the product $C \times \Lambda$, where $\Lambda = \mathbb{P}(H^0(L)^*)$. Let $p : C \times \Lambda \to C$ and $q : C \times \Lambda \to \Lambda$ be the two projections. Then $H^0(p^*L \otimes q^*\mathscr{O}_\Lambda(1)) = H^0(L) \otimes H^0(\mathscr{O}_\Lambda(1)) = H^0(L) \otimes H^0(L)^* = \mathrm{End}(H^0(L))$. Let $s \in H^0(p^*L \otimes q^*\mathscr{O}_\Lambda(1))$ be the section corresponding to the identity element in $EndH^0(L)$ and $D = \mathrm{div}(s)$ be the universal divisor. We then have the following exact sequence

$$0 \to \mathscr{O}_{C \otimes \Lambda} \to p^*L \otimes q^*\mathscr{O}_\Lambda(1) \to \mathscr{O}_D(D) \to 0.$$

Apply q_*, we have the exact sequence

$$0 \to \mathscr{O}_\Lambda \to H^0(L) \otimes \mathscr{O}_\Lambda(1) \to q_*\mathscr{O}_D(D) \to H^1(O_C) \otimes \mathscr{O}_\Lambda \to H^1(L) \otimes \mathscr{O}_\Lambda(1) \to 0,$$

because $D \to \Lambda$ is a finite morphism and $R^1 q_* \mathscr{O}_D(D) = 0$. We observe that the cockerel of $\mathscr{O}_\Lambda \to H^0(L) \otimes \mathscr{O}_\Lambda(1) \cong T_\Lambda$ and $q_*\mathscr{O}_D(D) \cong T_{C^{(g-1)}|\Lambda}$. (Why the second equality?????) So we obtain the following exact sequence

$$0 \to N \to H^1(O_C) \otimes \mathscr{O}_\Lambda \to H^1(L) \otimes \mathscr{O}_\Lambda(1) \to 0.$$

Hence we get $c_1(N^*) = c_1((\mathscr{O}_\Lambda(r+1)))$. Therefore,

$$\mathrm{mult}_{[L]}\Theta = c_1(\mathscr{O}_E(-E))^{g-2} = r + 1 = h^0(L).$$

(Why $(\psi^* c_1((\mathscr{O}_\Lambda(r+1))))^{g-2} = r + 1$????)

4. Singularities in higher dimension

Recall that a normal surface singularity (X, o) is Du Val, if there is a minimal resolution $f : Y \to X$ such that $K_Y \cdot E_i = 0$, equivalently, $K_Y = f^* K_X$.

The analogue in higher dimension is canonical singularities. Let $f : Y \to X$ be a log resolution, i.e. Y is smooth, f is proper, birational and the exceptional divisors are simple normal crossing. Write $K_Y - f^*K_X = \sum a_i E_i$, where a_i is called the log discrepancy of X along E_i. We say that X has canonical singularities, if all $a_i's$ are non-negative.

Theorem 4.1. *Assume that X is an algebraic variety with only canonical singularities and $f : Y \to X$ is a log resolution of X. Then*

$$\bigoplus_{m=0}^{+\infty} H^0(\mathscr{O}_X(mK_X)) = \bigoplus_{m=0}^{+\infty} H^0(\mathscr{O}_Y(mK_Y)).$$

Proof. Let E be the exceptional divisor. Notice that $f_*\mathscr{O}_E(E) = 0$. Apply f_* to the exact sequence

$$0 \to \mathscr{O}_Y \to \mathscr{O}_Y(E) \to \mathscr{O}_E(E) \to 0,$$

we see that $f_*\mathscr{O}_Y(E) = O_Y$. Apply the projective formula, we obtain the equality. $\square$

Definition 4.2 (Analytic version)**.** Let X be a smooth affine variety and $I_Z =< f_1, f_2, \ldots, f_r >$ be the ideal of a subvariety Z. For an positive number λ, the multiplier ideal of Z of weight λ is defined as

$$\mathscr{I}(I_Z^\lambda) = \{g \in \mathscr{O}_X \mid \frac{|g|^2}{(\sum |f_i|^2)^\lambda} \in \mathrm{L}^1_{loc}\}.$$

Example 4.3. Let $\mathscr{A} =< z^a >$ be an ideal in $k[z]$. The integral $\int |z^b| \mathrm{d}x\mathrm{d}y = \int r^b r \mathrm{d}r\mathrm{d}\theta$ exists if and only if $b + 1 > -1$. Therefore, $\mathscr{I}(^\lambda) =< z^{[\lambda a]} >$. More general, $\mathscr{I}((z_i^{a_1} \cdot z_n^{a_n})^\lambda) = ((z_i^{\lambda a_1} \cdot z_n^{\lambda a_n}))$.

Let $f : Y \to X$ be a log resolution of the pair (X, Z), where Z is a subvariety of X. The support $\mathrm{Supp}(Exc(f) + f^{-1}(Z)) = E_1 \cup E_2 \cup \cdots \cup E_r$ is simple normal crossing. Let $z_1, \cdots, z_n$ be local coordinates of X and $w_1, \cdots, w_n$ be local coordinates of Y. Then

$$\mathrm{d}w_1 \cdots \mathrm{d}w_n \mathrm{d}\bar{w}_1 \cdots \mathrm{d}\bar{w}_n = |Jac(f)|^2 dz_1 \cdots \mathrm{d}z_n \mathrm{d}\bar{z}_1 \cdots \mathrm{d}\bar{z}_n.$$

Write relative canonical divisor $K_{Y/X} = \operatorname{div}(\det(Jac(f))) = \sum k_i E_i$ and $I_z \cdot \mathscr{O}_Y = \mathscr{O}_Y(-\sum b_i E_i)$. Then $g \in \mathscr{I}(I_Z^\lambda)$ if and only if $\operatorname{ord}_{E_i} g \geq [\lambda b_i] - k_i$. So we can define multiplier ideal sheaves algebraically.

Definition 4.4 (Algebraic version)**.** Let $f : Y \to X$ be a log resolution of the pair (X, Z). Then the multiplier ideal sheaf of Z of weight λ is

$$\mathscr{I}(I_Z^\lambda) = f_* \mathscr{O}_Y(K_{Y/X} - [\lambda E]),$$

where E is the exceptional locus, i.e. $\mathscr{O}_Y(-E) = I_Z \cdot \mathscr{O}_Y$.

Definition 4.5. Let (X, D) be a pair where D is an effective $\mathbb{Q}$-divisor. The pair (X, D) is said to have

- Kawamata log terminal (kit) singularity if and only if the multipler ideal sheaf $\mathscr{I}(X, D) = \mathscr{O}_X$, equivalently, $k_i - b_i > -1$;
- terminal singularity if and only if $k_i - b_i > 0$;
- canonical singularity if and only if $k_i - b_i \geq 0$;
- log canonical singularity if and only if $k_i - b_i \geq 0$.

Theorem 4.6 (Kawamata-Viehweg vanishing theorem)**.** *Let (X, Δ) be a pair, where $\Delta = \sum \delta_i E_i$ is simple normal crossing and X is smooth. Let A be a nef and big $\mathbb{Q}$-divisor such that $K_X + A + \Delta$ is numerically equivalent to a line bundle L. Then $H^i(L) = 0$ for $i > 0$.*

Theorem 4.7 (Nadel vanishing theorem)**.** *Let X be a smooth variety and D be a $\mathbb{Q}$-divisor on X. Assume that L is an integral divisor such that $L - D$ is nef and big. Then*

$$H^i(\mathscr{O}_X(K_X + L) \otimes \mathscr{I}(D)) = 0, \text{ for all } i > 0,$$

where $\mathscr{I}(D)$ is the multiplier ideal sheaf of D.

Let A be an abelian variety of dimension g and Θ be a line bundle on A. Then the following are equivalent

(1) $c_i(\Theta)^g = g!$,

(2) $h^0(\Theta) = 1$,

(3) The morphism $\varphi_\Theta : A \to \mathrm{Pic}^0 A = A^*$ given by $\varphi_\Theta(x) = T_x\Theta \otimes \Theta^{-1}$, where T_x is the translation by x, is an isomorphism.

Definition 4.8. An abelian variety A together with a line bundle Θ satisfying one of the equivalent conditions is called a principal polarized abelian variety (p.p.a.v. for short), denoted by (A, Θ).

Theorem 4.9. *A p.p.a.v. (A, Θ) is log canonical if and only if the multiplier ideal sheaf $\mathscr{I}((1-\varepsilon)\Theta) = \mathscr{O}_A$ for any $\varepsilon > 0$.*

We need the following lemma.

Lemma 4.10. *Let (A, Θ) be a p.p.a.v. and Z be a closed subscheme of A. If $H^0(I_Z \otimes \Theta \otimes P) \neq 0$ for all $P \in \mathrm{Pic}^0(A)$, then $Z = \emptyset$.*

Proof. Since (A, Θ) is p.p.a.v., then $\Theta \otimes P = T_x\Theta$ for some $x \in A$. By the assumption, $H^0(I_Z \otimes \Theta \otimes P) \neq 0$, for all P. Therefore, $Z \in T_x\Theta$ for all $x \in A$. However, $\cap T_x\Theta = \emptyset$ which forces Z to be empty. □

Proof of Kollar's theorem. Assume for the contradiction that $\mathscr{I}((1-\varepsilon)\Theta) \neq O_A$ for some ε. Let Z be the subvariety such that the ideal sheaf $I_Z = \mathscr{I}((1-\varepsilon)$. It is clear that $H^0(I_Z \otimes \Theta) \neq 0$. Since Θ is nef and big, then $H^i(I_Z \otimes \Theta \otimes P) = 0$ for all $i > 0$ and $P \in \mathrm{Pic}^0(A)$ by Nadel vanishing theorem. Therefore, $\chi(I_Z \otimes \Theta) > 0$. Since $P \in \mathrm{Pic}^0(A)$, then $\chi(I_Z \otimes \Theta \otimes P) > 0$ which implies that $H^0(I_Z \otimes \Theta \otimes P) \neq 0$. So we see that $Z = \emptyset$. □

Theorem 4.11 (Ein-Lazarsfeld)**.** *Assume that (A, Θ) is a p.p.a.v. and Θ is irreducible. If Θ has canonical singularities then Θ has rational singularities.*

5. Adjoint linear systems on surfaces

Conjecture 5.1 (Fujita)**.** *Let X be a smooth projective variety of dimension n and A be an ample divisor on X. Then*

(1) $K_X + (n+1)A$ *is base-point-free.*

(2) $K_x + (n+2)A$ *is very ample.*

For surfaces, Reider proved the conjecture. The base-point-freeness of $K_X + (n+1)A$ in 3 and 4 dimensional was proved by Ein-Lazarsfeld and Kawamata respectively. The conjecture is open for higher dimensional varieties.

Reider's proof uses Bogomolov unstability theorem

Theorem 5.2 (Bogomolov)**.** *Let $\mathscr{E}$ be a rank 2 vector bundle on a smooth projective surface X. The the following are equivalent*

(1) $c_1^2(E) - 4c_2(E) > 0$.

(2) Let $L = \det(\mathscr{E})$. *There exists a divisor B, a 0-dimensional subscheme $W \subset X$ and an exact sequence*

$$0 \to \mathscr{O}_X(L - B) \to \mathscr{E} \to I_W \otimes \mathscr{O}_X(B) \to 0$$

such that $(L - 2B)^2 > 4 \deg W$ *and* $(L - 2B) \cdot H > 0$ *for any ample divisor H.*

For higher dimensional variety, so far we don't have any analogue of Bogomolov's theorem. The proofs of Ein-Lazarsfeld and Kawamata use multiplier ideal sheaves and Kawamata-Viehweg vanishing theorem.

Theorem 5.3 (Reider)**.** *Let X be a smooth projective surface and A be a nef and big divisor on X. Assume that $A^2 > 4$, then the linear system $|K_X + A|$ is base point free at a point $p \in X$, unless there is a curve B passing through p such that*

(1) $B^2 = -1$ *and* $(K_X + A) \cdot B = 0$, *or*

(2) $B^2 = 0$ *and* $(K_X + A) \cdot B = 1$.

Proof. Write $L = K_X + A$. Consider the exact sequence

$$0 \to \mathscr{O}_X(L) \otimes I_p \to \mathscr{O}_X(L) \to \mathscr{O}_X(L)|_p \to 0,$$

where I_p is the ideal sheaf of p in $\mathscr{O}_X$. Since A is nef and big, then $H^1(\mathscr{O}_X(L)) = 0$. The obstruction of $|L|$ being base point free at p is in $H^1(\mathscr{O}_X(L) \otimes I_p)$. By Serre duality, we have

$$(H^1(\mathscr{O}_X(L) \otimes I_p))^* \cong \mathrm{Ext}^1(\mathscr{O}_X(L) \otimes I_p, \mathscr{O})X(K_X)) = \mathrm{Ext}^1(\mathscr{O}_X(A) \otimes I_p, \mathscr{O}_X).$$

If $|L|$ is not base point free at p, then there is an nonzero element $\eta \in \mathrm{Ext}^1(\mathscr{O}_X(A) \otimes I_p, \mathscr{O}_X)$. So we have an extension

$$0 \to \mathscr{O}_X \to \mathscr{E} \to \mathscr{O}_X(A) \otimes I_p \to 0.$$

It is easy to check that $c_1(E)^2 - 4c_2(E) = A^2 - 4 > 0$. By Bogomolov theorem, we have an exact sequence

$$0 \to \mathscr{O}_X(A - B) \to \mathscr{E} \to \mathscr{O}_X(B) \otimes I_W \to 0,$$

such that $(A - 2B)^2 > 4$ and $(A - 2B) \cdot H > 0$ for any ample divisor H. Observe that the composition of morphism $\mathscr{O}_X(A - B) \to \mathscr{E} \to \mathscr{O}_X(A) \otimes I_p$ is nontrivial. So we see that $H^0(\mathscr{O}_X(B) \otimes I_p) \neq 0$. Then there is an effective divisor D linearly equivalent to B and passing through p. Since $c_2(\mathscr{E})) = 1$. Then $(A - D) \cdot D + \deg W = 1$. To prove the theorem, it suffices to show that $(A - D) \cdot D = 1$ and $D^2 \leq 0$. $\square$